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Equivalence between light-cone and conformal gauge in the two-dimensional string

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ABSTRACT

Aiming towards understanding the question of the discrete states in the light-cone gauge in the theory of two-dimensional strings with a linear background charge term, we study the path-integral formulation of the theory. In particular, by gauge fixing Polyakov's path-integral expression for the 2-d strings, we show that the light-cone gauge-fixed generating functional is the same as the conformal gauge-fixed one and is critical for the same value of the background charge (Q=2 $\sqrt{2}$). Since the equivalence is shown at the generating functional level, one expects that the spectra of the two theories are the same. The zero modes of the ratio of the determinants are briefly analyzed and it is shown that only the constant mode survives in this formulation. This is an indication that the discrete states may lie in these zero modes. This result is not particular to the light-cone gauge, but it holds for the conformal gauge as well.

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I. Introduction

The two-dimensional string theory with a linear background charge term is known to have a non-trivial spectrum, that is, an infinite number of physical states at discrete values of the momentum. These states, originally discovered in the $c = 1$ matrix model [1], have been well analyzed in this theory in the conformal gauge [2, 3, 4, 5, 6]. In the light-cone gauge, however, one naively expects that the two-dimensional theory should have no degrees of freedom left, other than the tachyon, since there are no transverse degrees of freedom in two dimensions and the light-cone gauge fixes completely the longitudinal components of the theory. Then the question which arises is how the light-cone gauge can see these extra states which are of longitudinal nature.

This is related to the question of the validity of the light-cone gauge itself. The main question we would like to address is the following: Is the light-cone gauge a physical gauge, in the sense that it gives the physical spectrum of the theory, or is it a non-legitimate choice which results in a truncation of the theory?

A related issue is the difference in the spectra obtained from BRST and other quantization procedures. It is widely agreed that BRST quantization gives the most trustworthy approach to the construction of the quantum string spectrum. However, one expects to obtain the same physical spectrum in the path-integral quantization procedure as well. All known studies on the subject have been done in the operator and BRST formalism of the theory. We are interested to study the problem in the path-integral formalism in an analogous fashion as Weinberg did for the vertex operators in string theories [7]. Specifically, we would like to observe how the discrete states are realized in Polyakov's path-integral formulation.

There have been two main works which address the question of the discrete states in the light-cone gauge in this theory. One is by Smith, who has fixed the light-cone gauge conditions $g_{--} = 0$ and $X^+ = p^+\tau$ independent of the conformal gauge [8]. He also fixes the g_{+-} component of the metric to one, using a modified Weyl symmetry (the pseudoWeyl symmetry) of the theory. He studied the BRST

cohomology of this light-cone gauge fixed theory and found no discrete states using this procedure.

The other independent approach is that of Rudd, who fixes the light-cone gauge condition after fixing the conformal gauge, using the residual symmetry left in the theory after imposing the conformal gauge conditions [9]. He needs to fix the X^+ component everywhere but at the special values of momentum in order to avoid singularities in the gauge fixing condition. In this way the corresponding modes of $X^\pm$ remain in the physical Hilbert space. He obtains a Lagrangian formulation of the gauge-fixed theory that explicitly depends on the oscillators (modes) of the X^+ component. He is using functional integration methods for his gauge fixing, but his analysis of the spectrum is based on the BRST charge operator.

In this paper we will investigate the light-cone gauge in Polyakov's path-integral formulation of the theory. The analysis will be exactly at the same level as in the conformal gauge. Specifically, we will fix only the two-dimensional reparametrization invariance using the light-cone gauge conditions, as in the conformal gauge, where one sees the discrete states before fixing the residual symmetry. The residual symmetry is interelated with the physical spectrum of the theory and fixing it from the start can overconstrain the system. The other point we would like to make is that we are not fixing the g_{+-} component of the metric at this point, the main reason being that the (pseudo)Weyl symmetry, used to fix this condition in the work of Smith, is not really a symmetry at the classical level of the theory. Therefore one does not know a priori how to treat a symmetry which is not classically there. Leaving the g_{+-} component free we can study the conformal invariance and anomaly of the theory in the path-integral formulation.

Before we start our analysis let's point out that the conformal gauge approach which is analyzed in the literature uses the one-dimensional string theory as its starting point. It has been shown that the one-dimensional non-critical string theory is equivalent to the two-dimensional critical string with a background charge of value equal to $2\sqrt{2}$ [10, 11]. In that approach the g_{+-} component of the metric

becomes the Liouville field which plays the role of an extra string dimension in the two-dimensional theory. It is not obvious, however, how to fix the light-cone gauge starting from the one-dimensional theory, since there are not even enough degrees of freedom to make a light-cone component. Moreover, this equivalence stems from the work of David, and Distler and Kawai [12, 13] which is based on a non-trivial anzatz. One does not know that similar arguments will give results in the light-cone gauge as well.

Therefore in this paper we study the light-cone gauge starting from the two-dimensional string with a background charge term, as in ref. [8]. Our main interest in this paper is to establish the equivalence between the light-cone and the conformal gauge fixing of the theory in the path-integral formulation of the theory. The light-cone gauge we use is in the same spirit as the original light-cone of Goddard, Goldstone, Rebbi and Thorn [14], that is, it is fixed independently of the conformal gauge [15].

The plan of this paper is as follows. In section II, we fix the light-cone gauge in the path-integral formulation of the theory. In section III, we quickly review the conformal gauge in the same formalism, in order to compare the results with the light-cone gauge ones. Section IV discusses the determinants in the light-cone gauge. (For completeness we give the details of the calculation of the determinant in an appendix.) It is shown explicitly that in this gauge the theory is conformally invariant for the value of the background charge $Q = 2\sqrt{2}$. Also by comparing the gauge-fixed expressions of the light-cone and conformal gauge we show the equivalence of the two theories at the generating functional level. Then we discuss the zero modes of the ratio of determinants in the light-cone gauge and their relevance to the discrete states. Finally, section V contains the conclusions.

II. Light-cone gauge fixing

We start from the following action for the two-dimensional string with a linear

background charge term:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma(\sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu + \alpha' n \cdot X \sqrt{g} R^{(2)}) \tag{1}$$

where $\sigma = (\sigma^0, \sigma^1)$ are coordinates in the two-dimensional parameter space. $X^\mu(\sigma)$, $\mu = 1, 2$ defines a map from the two-dimensional parameter space to a surface in the two-dimensional physical space-time. The metric g_{ab} characterizes the parameter space. $n^\mu = (n^1, 0)$ is a space-time vector, α' is the inverse string tension and $R^{(2)}$ is the Ricci scalar.

This action is invariant under two-dimensional reparametrizations while the Weyl symmetry is broken by the background charge term. One can recover the Weyl symmetry, however, by scaling the X^μ field accordingly, that is, by making the following transformations:

$$\begin{aligned} \delta g_{ab} &= \epsilon g_{ab} \\ \delta X^\mu &= -\frac{\alpha' n^\mu \epsilon}{2} \\ \delta R^{(2)} &= -\epsilon R^{(2)} - \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} \partial_b \epsilon) \ . \end{aligned} \tag{2}$$

It is easy to see that this transformation leaves the classical action invariant only to order α'. Terms of order ${\alpha'}^2$ break this symmetry. Despite the fact that this is not a symmetry of the classical action it has been used in the literature as such in order to fix the third component of the metric [8, 9]. In this paper we will take the point of view that this is not a symmetry of our action and therefore we will not fix it. In order to make our argument more clear, we will analyze both the conformal and light-cone gauge in parallel. The gauge conditions will be fixed in both cases using the reparametrization invariance of the action. Our formulation is Polyakov's path-integral one. We also start with a general background charge which will be specified later.

In the rest of the paper, mainly for calculational convenience, but also because it makes our discussion more transparent, we will use the following redefined field

variables, as in [8]

$$\begin{aligned}\tilde{g}^{ab} &= \sqrt{g}g^{ab} \\ \tilde{X}^{\mu} &= X^{\mu} + \frac{\alpha' n^{\mu}}{2}\ln\sqrt{g}\ .\end{aligned} \tag{3}$$

The new fields are invariant under the Weyl transformations (2). In terms of these new variables the action takes the following form

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma[\tilde{g}^{ab}\partial_a\tilde{X}_{\mu}\partial_b\tilde{X}^{\mu} + \alpha' n\cdot\tilde{X}\tilde{R}^{(2)} - \frac{{\alpha'}^2 n^2}{2}\ln\sqrt{g}(\tilde{R}^{(2)} - \frac{1}{2}\tilde{\Delta}\ln\sqrt{g})] \tag{4}$$

where $\tilde{R}^{(2)} = \sqrt{g}R^{(2)} + \sqrt{g}\Delta\ln\sqrt{g}$ and $\tilde{\Delta} = \partial_a(\tilde{g}^{ab}\partial_b)$. Notice that the action (1) cannot be re-expressed completely in terms of the Weyl invariant fields (3). The presence of the $\sqrt{g}$ in the terms of order ${\alpha'}^2$ in (4) reflects the fact that the action is not invariant under Weyl transformations.

Polyakov's path-integral formulation of the Virasoro amplitude is given by

$$\int\cdots\int\prod_j d^2\sigma_j\sqrt{g}(\sigma_j)\mathcal{D}g_{ab}\mathcal{D}\tilde{X}^{\mu}e^{iS\ +\ i\int d^2\sigma j^{\mu}\tilde{X}_{\mu}} \tag{5}$$

where S is given by (4) and j^{μ} is an external source term defined by

$$j^{\mu}(\sigma) = \sum_j \delta^2(\sigma - \sigma_j)k_j^{\mu}\ . \tag{6}$$

The reparametrization-invariant measure in (5) is defined by

$$\begin{aligned}\mathcal{D}g_{ab} &= \prod_{\sigma} dg_{++}dg_{+-}dg_{--}(g)^{-3/2} \\ \mathcal{D}X^{\mu} &= \prod_{\sigma} dX^{+}dX^{-}\ .\end{aligned} \tag{7}$$

(Notice that from now on we drop the tilde notation from the X^{μ} variable in order to simplify the notation.) We use the Minkowski metric for the parameter space. In writing the integral representation of Virasoro amplitude (5), we assume

that we have divided by the Ḿobius volume and the reparametrization volume (diffeomorphism), since the integrand is invariant under two-dimensional general coordinate transformations.

We choose the light-cone gauge conditions as in ref. [15], that is,

$$\begin{aligned} g_{--} &= 0 \\ X^{+} &= f(\sigma)\ . \end{aligned} \tag{8}$$

The first of these conditions corresponds to $(\dot{X} - X')^2 = 0$ in the Nambu-Goto Lagrangian and $f(\sigma)$ is an arbitrary function of the two-dimensional σ. This is the gauge originally chosen by Gervais and Sakita in their study of the interacting string [16].

We fix the gauge using the invariance of the action under the general coordinate transformations in two-dimensional parameter space

$$\sigma^a \to \sigma^{(\epsilon)a} = \sigma^a + \epsilon^a(\sigma)\ . \tag{9}$$

Under infinitesimal reparametrization transformations the variables X^μ and g_{ab} transform as

$$\begin{aligned} X^\mu \to X^{(\epsilon)\mu}(\sigma) &= X^\mu(\sigma) - \epsilon^a \partial_a X^\mu(\sigma) - \frac{\alpha' n^\mu}{2} \partial_a \epsilon^a \\ g_{ab} \to g^{(\epsilon)}_{ab} &= g_{ab} - \epsilon^c \partial_c g_{ab} - \partial_a \epsilon^c g_{cb} - \partial_b \epsilon^c g_{ac}\ . \end{aligned} \tag{10}$$

According to the Faddeev-Popov procedure, we change the integration variables dX^+ and dg_{--} to the two parameters of the transformations $d\epsilon^+$ and $d\epsilon^-$. The light-cone notation that we use throughout this paper is

$$\begin{aligned} \sigma^{\pm} &= \frac{1}{\sqrt{2}}(\sigma^0 \pm \sigma^1) \\ g_{\pm\pm} &= \frac{1}{2}(g_{11} + g_{00} \pm 2 g_{01}) \\ g_{+-} &= \frac{1}{2}(g_{11} - g_{00})\ . \end{aligned} \tag{11}$$

Next we insert in the integral the identity

$$\int D\epsilon\ \Delta_{FP}(X,g) \prod_\sigma \delta(X^{+(\epsilon)} - f(\sigma)) \prod_\sigma \delta(g_{--}^{(\epsilon)}) = 1 \tag{12}$$

where $D\epsilon$ denotes the invariant measure of the transformation and Δ_{FP} is the Faddeev-Popov determinant given by

$$\Delta_{FP} = \det \begin{pmatrix} -\partial_+ f - \frac{\alpha' n^+}{2}\partial_+ & -\partial_- f - \frac{\alpha' n^-}{2}\partial_- \\ -2g_{+-}\partial_- & 0 \end{pmatrix} . \tag{13}$$

The next step is to make an inverse reparametrization transformation to all variables in the path-integral. Since the integrand, measure and Faddeev-Popov determinant are invariant under the transformation, the integration over the parameter ϵ can be factored out as a constant equal to the volume of the reparametrization group.

The integrations over X^+ and g_{--} are trivially carried out and the gauge fixed action is given by

$$\begin{aligned} S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \big[& 2\tilde{g}^{--}\partial_- X^- \partial_- f + 4\partial_+ f \partial_- X^- - \alpha' n^+ X^- \partial_-^2 \tilde{g}^{--} \\ & -\alpha' n^+ f \partial_-^2 \tilde{g}^{--} + \alpha'^2 n^{+2} \rho [\partial_-^2 \tilde{g}^{--} + \partial_+\partial_-\rho + \frac{1}{2}\partial_-(\tilde{g}^{--}\partial_-\rho)] \Big] \\ & + i \int d^2\sigma \, \big[j^+ X^- + j^- f(\sigma) \big] \end{aligned} \tag{14}$$

where we have used the fact that $n^- = n^+$ and ρ stands for $\ln\sqrt{g}$. Notice also that $g^{+-} = -\frac{1}{|g|} g_{+-}$ and $\tilde{g}^{+-} = \sqrt{g} g^{+-} = 1$ in this gauge.

Next we choose the function $f(\sigma)$ such that it satisfies the following equation:

$$\partial_-\partial_+ f(\sigma) = -\pi\alpha' j^+(\sigma) \ . \tag{15}$$

With this choice of $f(\sigma)$ the linear terms in X^- in (14) cancel. The solution of

(15) is given by

$$f(\sigma) = \alpha' \sum_j k_j^+ \ln|\sigma - \sigma_j| \ . \tag{16}$$

The integration over X^- in the integral (5) is now easily carried out and results in the following inverse determinant and delta function for $\tilde{g}^{--}$

$$\left[det' \left(\partial_- f \partial_- - \frac{\alpha' n^+}{2} \partial_-^2 \right) \right]^{-1} \delta(\tilde{g}^{--}) \tag{17}$$

where by prime we mean that the determinant does not contain the zero mode. Because of the delta function we can now perform the integration over g_{++}. We obtain

$$\int \cdots \int \prod_k d^2\sigma_k d\rho \frac{\Delta_{FP}}{det(\partial_- f \partial_- - \frac{Q}{2\sqrt{2}} \partial_-^2)} e^{\frac{iQ^2}{8\pi\alpha'} \int d^2\sigma \partial_+\rho\partial_-\rho - \sum_i \sum_{j\neq i} k_i^- k_j^+ ln|\sigma_i - \sigma_j|} \tag{18}$$

where we have substituted e^ρ for g_{+-} and Q for $\sqrt{2}\alpha' n^+$. From now on we omit the prime from the determinant. The exclusion, however, of the zero modes in the last ratio of determinants is to be understood.

The last expression is the light-cone gauge-fixed path-integral that will concern us in the rest of this paper. In this expression, however, all variables and determinant operators are defined on the two-dimensional world sheet. In order for this to coincide with Mandelstam's formulation of strings, one has to perform Mandelstam's mapping [17]. This procedure is well known and analyzed in a similar context in ref. [15], but for completeness we shall give at this point a brief discussion for this specific case.

We first have to perform the following Wick rotation

$$\sigma^0 = i\xi^2 \quad , \qquad \sigma^1 = \xi^1 \ . \tag{19}$$

Under this rotation the path-integral becomes

$$\int \cdots \int \prod_k d^2 z_k d\rho \frac{\Delta_{FP}}{det(\partial_{\bar{z}} f \partial_{\bar{z}} - \frac{Q}{2\sqrt{2}} \partial_{\bar{z}}^2)} e^{\frac{Q^2}{8\pi\alpha'} \int d^2\xi \partial_z\rho\partial_{\bar{z}}\rho - \sum_{i\neq j} k_i^- Re\mathcal{F}(z_i)} \tag{20}$$

where

$$\mathcal{F}(z) = \sum_j k_j^+ ln(z - z_j) \tag{21}$$

and z stands for z and $\bar{z}$ since $z = \xi^1 + i\xi^2$ and $\bar{z} = \xi^1 - i\xi^2$. In this expression the variables ξ and z are defined on the entire complex plane.

Then we make Mandelstam's conformal transformation

$$z \rightarrow w = \mathcal{F}(z) \equiv \tau + i\sigma \ . \tag{22}$$

This defines a map from the entire plane to Mandelstam's tube. Under this transformation our gauge fixed function $f(\sigma)$ becomes the usual light-cone gauge condition τ. The points z_i transform into strings i at initial and final times $\tau = \mp\infty$, depending on the values of k_i^+ , $k_i^+ > 0$ or $k_i^+ < 0$ for incoming or outgoing strings, respectively. The last exponential factor of (18) becomes

$$exp(-\sum_i k_i^- \tau_i) \ . \tag{23}$$

where $\tau_i = \sum_{j \neq i} k_j^+ ln|z_i - z_j|$. Then we perform the same transformation on the ρ integration variable and the determinants such that all variables are defined on Mandelstam's tube. The result for the path-integral is now

$$\int \cdots \int \prod_k d\tilde{\tau}_k d\rho \frac{\Delta_{FP}}{det(\partial_{\bar{w}} f \partial_{\bar{w}} - \frac{Q}{2\sqrt{2}} \partial_{\bar{w}}^2)} e^{\frac{Q^2}{8\pi\alpha'} \int d^2 w \partial_w \rho \partial_{\bar{w}} \rho - \sum_i k_i^- \tau_i} \tag{24}$$

where $\tilde{\tau}_k$ are the interaction times of the string. Note that the exponent (23) does not depend on the variable w and therefore comes out of the w-integral. However, this exponential factor implicitly depends on the interaction times $\tilde{\tau}_k$ since changing τ_i reflects a change in both τ_i and $\tilde{\tau}_i$, which are defined by

$$\tilde{\tau}_i = \mathcal{F}(\tilde{z}_i), \quad \text{where} \quad \frac{\partial \mathcal{F}(z)}{\partial z}|_{z=\tilde{z}_i} = 0. \tag{25}$$

Thus the integration over the $\tilde{\tau}_k$ variables makes this factor non-trivial and therefore the dependence of the path-integral on both k^- and k^+ essential.

The expression (24) gives Mandelstam's picture for two-dimensional strings with a background charge. Since there are no transverse degrees of freedom in this theory, the exponent is essentially the factor (23). The determinants depend on the g_{+-} component of the metric, which is a result of the non-invariance of the measure under conformal symmetry. The g_{+-} (or otherwise ρ)-dependence of the integrand defines the conformal anomaly. In this theory, due to the fact that the background charge is a quantum correction of the classical action, there is also a ρ-dependence in the exponent of the integrand and hence an extra contribution to the conformal anomaly due to this term. A straightforward way to see this is by using the action (4). The change of the action under Weyl transformation gives essentially the result

$$\int d^2\sigma\sqrt{g}R^{(2)}\epsilon \tag{26}$$

which is nothing but the Weyl anomaly term. In this sense the background term charge plays the role of an effective (WZW-like) action for the conformal anomaly [18]. This fact will be explicit from our calculations below.

The anomaly in the light-cone gauge will be given in section IV. In the next section, in order to compare our results with the results of the conformal gauge, we briefly analyze the conformal gauge in this theory.

III. Conformal gauge fixing.

We start again from Polyakov's path-integral expression, given by (5), and we fix the two-dimensional reparametrization invariance of the action, as before.

The conformal gauge conditions are now

$$g_{++} = 0 \quad \text{and} \quad g_{--} = 0$$

where, again, we use light-cone notation.

We use the Faddeev-Popov method for the gauge fixing. The Faddeev-Popov

determinant is now an essentially diagonal matrix given by

$$\Delta_{FP} = \begin{pmatrix} 0 & -2g_{+-}\partial_+ \\ -2g_{+-}\partial_- & 0 \end{pmatrix} . \tag{27}$$

After trivially integrating over g_{++} and g_{--} in the path-integral, we obtain

$$\begin{aligned} \int \cdots \int \prod_k d^2\sigma_k \sqrt{g}(\sigma_k) d\rho dX^+ dX^- \Delta_{FP} e^{-\frac{i}{\pi\alpha'}\int d^2\sigma \left(\partial_+ X^+ \partial_- X^- - \frac{Q^2}{8}\partial_+\rho\partial_-\rho\right)} \\ e^{i\int d^2\sigma (j^+X^- + j^-X^+)} . \end{aligned} \tag{28}$$

This is the path-integral expression for the conformal gauge fixed theory. The source term is defined again by (6). In order to obtain the determinants and extract the ρ-dependence of the part of the action containing the matter fields, we perform the integrations over X^+ and X^-. Because of the presence of the source term, the integrations are done as follows: we first integrate over the X^- field. The result is the following delta function

$$\delta(\partial_+\partial_- X^+ + \pi\alpha' j^+) . \tag{29}$$

This, then, means that the integration over X^+ replaces the X^+ variable in the action with the solution of the following equation

$$\partial_+\partial_- X^+ = -\pi\alpha' j^+(\sigma) \tag{30}$$

and gives the following inverse determinant

$$[det(\partial_+\partial_-)]^{-1} . \tag{31}$$

Comparing at this point (30) with the equation (15) of the previous section, we see that the variable X^+ of this gauge satisfies the same equation as the function $f(\sigma)$ of the light-cone gauge conditions. The meaning of this is that after the integration over X^-, the X^+ field of the conformal gauge-fixed theory takes the value of the light-cone gauge fixed theory.

Then the expression (28) becomes

$$\int \cdots \int \prod_k d^2\sigma_k d\rho \frac{\Delta_{FP}}{det(\partial_+\partial_-)} e^{\frac{iQ^2}{8\pi\alpha'} \int d^2\sigma(\partial_+\rho\partial_-\rho) \; - \; \sum_{i\neq j} k_i^- f(\sigma_i)} \tag{32}$$

where in the last expression we have called $f(\sigma)$ the solution of the equation (30) in an obvious notation.

Comparing the last expression with (18) we see that they are the same except for the determinant-ratio factor. The calculation of the determinants of the operators in the conformal gauge is known. Therefore, we only state the results. They are given by

$$det(\partial_+\partial_-) = e^{-\frac{i}{12\pi\alpha'} \int d^2\sigma\partial_+\rho\partial_-\rho} \tag{33}$$

$$\Delta_{FP} = e^{-\frac{26i}{24\pi\alpha'} \int d^2\sigma\partial_+\rho\partial_-\rho} \; . \tag{34}$$

Putting now (32) together with (33) and (34) we see that the anomaly in the conformal gauge cancels for the value of the background charge $Q = 2\sqrt{2}$.

In the next section we shall discuss these determinants in the light-cone gauge. We shall show that the contribution to the anomaly coming from the determinants and the contribution coming from the background charge term again cancel for the same value of the background charge, that is, for $Q = 2\sqrt{2}$.

IV. The Determinant in the light-cone gauge

The Faddeev-Popov determinant in the light-cone gauge and the determinant coming from integrating out the matter field variables are combined in the following ratio

$$\frac{det \begin{pmatrix} -x_+ - \frac{Q}{2\sqrt{2}}\partial_- & -x_- - \frac{Q}{2\sqrt{2}}\partial_- \\ -2g_{+-}\partial_- & 0 \end{pmatrix}}{det \left(x_-\partial_- - \frac{Q}{2\sqrt{2}}\partial_-^2 \right)} \tag{35}$$

where we have denoted $\partial_- f$ by x_-.

In order to calculate this determinant we need to regularize the functional integrations associated with it and extract the ρ dependence from the ratio (35) . We use the Pauli-Villars regularization procedure [19]. Since this method is well analyzed in ref. [14], we here state only the result. (For completeness we include the calculations of the determinant in the appendix A.) It is given by

$$\frac{\Delta_{FP}}{\det(x_- \partial_- - \frac{Q}{2\sqrt{2}} \partial_-^2)} = exp\left[-\frac{24}{24\pi\alpha'} \int d^2\xi [\frac{1}{2}(\partial_+ \rho \partial_- \rho) + \mu^2 e^{\rho}] \right] \tag{36}$$

Due to the form of the determinant in the light-cone gauge, one, naively, would expect that the result depend explicitly on Q. However, because of cancelations between the numerator and denominator the result does not depend on the background charge Q.

It is now easy to see that the total ρ-dependence on the integrand in the expression (24) cancels for the value of the background charge $Q = 2\sqrt{2}$.

The conclusions from the last results are: first, that the two-dimensional string theory with a background charge is conformally (scale) invariant at the quantum-mechanical level for the specific value of the background charge $Q = 2\sqrt{2}$. Even if this result has been known in the conformal gauge for some time (see for instance ref. [10, 13]), in this paper we have made it more clear in both gauges†. We have explicitly shown that the background charge term in the original action is nothing but the Wess-Zumino term for the anomaly, since the contribution coming from that term cancels exactly the one coming from the ratio of determinants for the specific value of Q. The second conclusion obtained from the above analysis is that the light-cone gauge-fixed 2-d string theory is equivalent to the conformal gauge-fixed one. The path-integral expressions (18) and (32) coincide except for the ratio of determinants. In fact, since we have integrated out all the fields except ρ, in our approach, this determinant factor is essentially the content of the theory. In this

† A light-cone gauge canonical approach has been also analyzed by Myers in the theory of d-dimensional string with background charge term [20].

section we showed that the determinants coincide as well, showing the complete equivalence of the two gauge-fixed theories.

Then, since the two theories have been shown to be equivalent at the partition function level, one would conclude that the spectra of the theories are the same. Therefore the light-cone gauge must contain the physical states that are present in the conformal gauge. In the path-integral approach, however, naively it seems that, after integrating out the $X^{\pm}$ variables and calculating the determinant, nothing which has some physical content is left in the theory. See, for instance, the expression (24) in the light-cone gauge. Therefore any physical states contained in the theory must originate from the factor $\exp(\sum_i k_i^- \tau_i)$ and the zero modes of the determinant. (Remember that the zero modes of the determinant were excluded in our previous discussion.) We shall briefly discuss the zero modes of the determinants next.

The zero modes of the Faddeev-Popov determinant correspond to the residual gauge transformations left in the theory after fixing the reparametrization invariance. These are given by

$$\begin{aligned} \epsilon^+ &= \epsilon^+(\sigma^+) \\ \epsilon^- &= -(\epsilon^+ + 2\epsilon'^+) + \tilde{g}(\sigma^+)e^{-\frac{1}{2}\sigma^-} \end{aligned} \tag{37}$$

where by prime we mean derivative with respect to the argument and the $\tilde{g}$ here is an arbitrary function of σ^+ only.

On the other hand the zero modes of the determinant in the denominator correspond to the unphysical degrees of freedom and are given by

$$\Psi = 2\pi(\sigma^+) + \phi(\sigma^+)e^{-\frac{1}{2}\sigma^-} \tag{38}$$

where π and ϕ are arbitrary functions of the argument. These "residual" degrees of freedom are the components of the gauge field which change under the residual transformations and are eliminated upon fixing the residual gauge transformation.

If the theory contained no discrete states, they should be exactly as many as the number of the residual gauge transformations. (In practice the elimination of the unphysical degrees of freedom is taking place by imposing on the states the physicality condition, which is the generator of the residual symmetry.) In this case, however, since extra states exist, there exist some residual gauge transformations which do not change the gauge fields. These gauge transformations do not eliminate unphysical degrees of freedom. Therefore for the corresponding values of momenta extra states remain in the theory. (In the case of QED, for instance, the only degree of freedom which remains this way in the theory is the trivial constant mode, which exists at zero momentum.) In terms of the zero modes of the determinants, then, the expectation is that the Faddeev-Popov determinant should have some zero modes which do not correspond to zero modes of the determinant in the denominator. These extra zero modes are at the momenta at which the discrete states exist.

Comparing, however, the solutions of (38) to the ones of (37) we find that the only residual gauge transformation which does not eliminate unphysical degree of freedom in this case corresponds again to the constant mode. A similar calculation in the conformal gauge gives exactly the same result. Only the constant mode survives from the determinants there as well. This result is expected, since in our path-integral formalism we have included only the tachyon vertex operator. The discrete states corresponding to higher spin particles should arise in a similar way by including the appropriate vertex operators in the path-integral. Our present result, therefore, provides an indication that the discrete states may lie in the zero modes of the determinants, and this is not particular to the light-cone gauge. This is one main observation in this paper.

V. Conclusions

In this paper we have studied the light-cone gauge in two-dimensional string theory with a linear background charge term in the path-integral formulation. One

of the aims of this work is to understand the presence of the background charge in the Lagrangian and the role of it in the conformal invariance of the theory in light-cone gauge. This question becomes relevant since classically the background charge term breaks Weyl symmetry. One therefore would like to understand how the conformal symmetry is restored at the quantum mechanical level. We have shown that one can fix the reparametrization invariance by choosing the light-cone gauge conditions in this theory independently of the conformal gauge. The gauge-fixed theory then depends on the conformal component of the metric and in general it is not conformally invariant. By requiring conformal invariance of the theory at the quantum-mechanical level, we have shown that the conformal anomaly cancels when the value of the background charge equals $2\sqrt{2}$. In particular, the dependence on the conformal factor coming from the ratio of determinants cancels the dependence coming from the background charge term exactly at that value of the background charge. This is an explicit realization of the fact that the background charge term plays the role of the Wess-Zumino term in the Lagrangian [18].

Moreover we have shown that the light-cone gauge-fixed theory obtained by imposing the light-cone conditions from the beginning, that is, independently of the conformal gauge, is equivalent to the theory one obtains in conformal gauge†. This has been shown by comparing the two path-integral expressions after fixing the conformal and light-cone gauges respectively. The calculation of the determinants in both gauges give the same result.

Finally, let's emphasize that the ultimate goal of this approach is to understand the existence of the discrete states in the light-cone gauge in the path-integral formulation. This question has not yet been answered completely in this paper. Since, however, the conformal gauge-fixed and the light-cone gauge-fixed theories are equivalent at the partition function level one expects that the spectra of the two theories are the same. Specifically, in our path-integral formulation the whole

† Similar results, which show the equivalence between the conformal gauge and Polyakov's light-cone gauge in this theory, have been recently obtained by Kato using a similarity transformation in the BRST charge [22].

theory is contained in the ratio of the determinants and an exponential factor which depends explicitly on the external momenta. From our analysis of the zero modes of the determinants we have concluded that only the constant mode is hidden in the zero modes of the determinant in this formalism, which may correspond to the tachyon vertex operator present in our theory. This result is not a peculiarity of the light-cone gauge. The conformal gauge path-integral gives exactly the same results.

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Appendix A. Calculation of the determinants

In this appendix we discuss the calculation of the ratio of determinants in the light-cone gauge. Due to the fact that our gauge fixing is not diagonal, the determinant operators are non-diagonal in this gauge. The components of the Faddeev-Popov determinant do not have the same signature. They both operate on vector fields but give back scalars and two-tensors respectively. In order to reduce the determinants to those of Laplacian (self-adjoint) operators we use a conformal invariant† regularization.

The best way to understand the ρ-dependence in the determinants is by using Pauli-Villars regularization [19]. According to this procedure, the regularization of the operator ∇^2_ξ is done as follows: A number of auxiliary Bose or Fermi fields, with masses M_i, are introduced in order to cut off the large momentum contribution.

† By "conformal invariance" here we mean the symmetry which transforms analytic functions into analytic and anti-analytic into anti-analytic.

The regularized determinant is then the determinant due to all these fields:

$$\prod_i [\det(\nabla_\xi^2 - M_i^2 e^{\rho(\xi)})]^{C_i} \tag{39}$$

where $C_i = \pm 1$ depending on the statistics, with $C_0 = 1$ and $M_0 = 0$. The multiplication of the mass term with $e^{\rho(\xi)}$ is done in order to have conformally invariant operators. At the end of the calculation we set

$$1 + \sum_i C_i = 0, \quad \sum_i C_i M_i^2 = 0, \quad \sum_i C_i M_i^2 \ln M_i \Big|_{M_i \to \infty} = \text{finite}, \tag{40}$$

to assure that the result will be finite (regularized). The ρ-dependence of the determinant is then computed by

$$\frac{\delta}{\delta\rho(\xi)} \ln\det \nabla_\xi^{2\,reg} = -\sum_i C_i M_i^2 e^{\rho(\xi)} < \xi | \frac{1}{\nabla_\xi^2 - M_i^2 e^{\rho(\xi)}} | \xi > \ . \tag{41}$$

In order to correctly calculate the quotient determinant (35), one has to regularize the ratio of determinants such that the infinities between the numerator and denominator cancel. Also, the Faddeev-Popov determinant is an infinite by infinite matrix and multiplication of the determinants associated with these operators is not allowed in principle. It is, however, an intractable problem to regularize the Faddeev-Popov determinant as it is. In the calculation which follows we regularize the two operators of the Faddeev-Popov determinant separately. In order to justify this we use the action formulation.

Let's express the Faddeev-Popov determinant in terms of ghost and antighost fields. It is given by

$$\Delta_{FP} = \int db\, db^{--} dc^+ dc^- e^{-i\int d^2\sigma \sqrt{g}\left[b(x_- + \frac{Q}{2\sqrt{2}}\partial_-)c^- + 2b^{--} g_{+-}\partial_- c^+ \right]} \ . \tag{42}$$

Now we observe that the two terms in the action of (42) are expressed in terms of two independent sets of fields, i.e. b^{--}, c^+ and b, c^-. Therefore they can

be regularized in the action independently, that is, by introducing two different Pauli-Villars terms. This is equivalent to regularizing the determinants of the two operators in the numerator of (35) separately. In what follows we shall regularize the $\det(-2g_{+-}\partial_-)$ and $\det(-x_- - \frac{Q}{2\sqrt{2}}\partial_-)$ independently.

On the other hand, the bosonic part of the determinant can be viewed as two independent determinants, i.e. $\det(\partial_-)$ and $\det(x_- - \frac{Q}{2\sqrt{2}}\partial_-)$. Again this stems from the fact that the relevant part of the action can be written in first-order formalism by introducing auxiliary fields. It is

$$\tilde{g}^{--}(x_-\partial_- - \frac{Q}{2\sqrt{2}}\partial_-^2)X^- = \phi^{--}\partial_-\xi_- + \psi(x_- + \frac{Q}{2\sqrt{2}}\partial_-)\lambda^- \tag{43}$$

where ϕ^{--}, ξ_-, λ^- and ψ are auxiliary fields. Notice here that the action in first-order formalism has been written in such a way that the operators involved act on spaces with tensorial signatures similar to the ones of the ghost action. One also finds that $\det(x_- + \frac{Q}{2\sqrt{2}}\partial_-) = \det(x_- - \frac{Q}{2\sqrt{2}}\partial_-)$, since if $\psi(x)$ is an eigenfunction of the first operator with eigenvalue λ, then $\psi(-x)$ is an eigenfunction of the second operator with the same eigenvalue λ and the same boundary conditions. Therefore, both operators have the same eigenvalues and hence the same determinant. In what follows we regularize the bosonic and ghost-term operators using the method described above.

We first compute the adjoints of the operators of the determinants. The determinants of the interesting operators are then computed by extracting the square root of $\det O^\dagger O$, where O stands for the relevant operator. In order to compute the adjoints of the operators we must define appropriate inner products. We define our inner products such that they are conformally covariant. (We use the g_{+-} component of the metric in order to lower and raise the indices.) The invariant inner product of two-tensor fields is defined by

$$(\Psi_{++}, X_{++}) = \int d\sigma^+ d\sigma^- g^{+-}\Psi^*_{++}X_{++}, \tag{44}$$

where $\Psi^*_{++} = \Psi_{--}$, while the inner products of two-scalars and two-vectors are

defined by

$$(\Psi, X) = \int d\sigma^+ d\sigma^- g_{+-} \Psi^* X \tag{45}$$

and

$$(\Psi^+, X^+) = \int d\sigma^+ d\sigma^- g_{+-}^2 \Psi^{+*} X^+ \tag{46}$$

respectively.

For the operators of the Faddeev-Popov determinant we have: If $P_1 = -x_- - \frac{Q}{2\sqrt{2}}\partial_-$ the adjoint $P_1^\dagger$ is defined by

$$(\delta X, P_1 \delta\epsilon^-) = (P_1^\dagger \delta X, \delta\epsilon^-). \tag{47}$$

Using the last relation and the definitions for the inner products we obtain

$$P_1^\dagger = {g^{+-}}^2(-x_+ + \frac{Q}{2\sqrt{2}}\partial_+)g_{+-}. \tag{48}$$

For $P_2 = -2g_{+-}\partial_-$ the adjoint $P_2^\dagger$ is defined by

$$(\delta g_{--}, P_2 \delta\epsilon^+) = (P_2^\dagger \delta g_{--}, \delta\epsilon^+). \tag{49}$$

It is given by

$$P_2^\dagger = 2{g^{+-}}^2\partial_+. \tag{50}$$

Similarly, we compute the adjoints for the operators in the denominator of (35). For $T_1 = x_- + \frac{Q}{2\sqrt{2}}\partial_-$ the adjoint is $T_1^\dagger = -{g^{+-}}^2(-x_+ + \frac{Q}{2\sqrt{2}}\partial_+)g_{+-}$. For $T_2 = \partial_-$, the adjoint is $T_2^\dagger = -\partial_+ g^{+-}$.

The dependence of these operators on ρ is computed by taking the variation of $\ln\det P_1^\dagger P_1$, $\ln\det P_2^\dagger P_2$, $\ln\det T_1^\dagger T_1$ and $\ln\det T_2^\dagger T_2$ where

$$P_1^\dagger P_1 = e^{-\rho}(x_+ - \frac{Q}{2\sqrt{2}}\partial_+ - \frac{Q}{2\sqrt{2}}\partial_+\rho)(x_- + \frac{Q}{2\sqrt{2}}\partial_-) \tag{51}$$

and

$$P_2^\dagger P_2 = -4e^{-2\rho}\partial_+(e^\rho \partial_-) \tag{52}$$

while for the T operators it is

$$T_1^\dagger T_1 = e^{-\rho}(x_+ - \frac{Q}{2\sqrt{2}}\partial_+ - \frac{Q}{2\sqrt{2}}\partial_+\rho)(x_- + \frac{Q}{2\sqrt{2}}\partial_-) \tag{53}$$

and

$$T_1^\dagger T_1 = -\partial_+(e^{-\rho}\partial_-) \tag{54}$$

where we have substituted e^ρ for g_{+-}.

Then the variation of the whole ratio of the determinants is given by

$$\begin{aligned} &\delta \ln \frac{\sqrt{\det}(P_1^\dagger P_1)\sqrt{\det}(P_2^\dagger P_2)}{\sqrt{\det}(T_1^\dagger T_1)\sqrt{\det}(T_2^\dagger T_2)} = \\ &\frac{1}{2}\left(\delta \operatorname{tr}\ln(P_1^\dagger P_1) + \delta \operatorname{tr}\ln(P_2^\dagger P_2) - \delta \operatorname{tr}\ln(T_1^\dagger T_1) - \delta \operatorname{tr}\ln(T_2^\dagger T_2)\right) \end{aligned} \tag{55}$$

Now, since the operators $P_1^\dagger P_1$ and $T_1^\dagger T_1$ are the same, they cancel out of the last expression and the variation of the ratio of the determinants is determined only by the variation of $P_2^\dagger P_2$ and $T_2^\dagger T_2$.

Following the regularization procedure given by (39) and (40), we write the regularized quantities to be computed as

$$F_1 = \sum_i C_i tr \ln(A_1 + M_i^2 e^\rho) \tag{56}$$

and

$$S_1 = \sum_i C_i tr \ln[B_1 + M_i^2 e^\rho] \tag{57}$$

where $A_1 = \partial_+\partial_- + \partial_+\rho\partial_-$ and $B_1 = (\partial_+\partial_- - \partial_+\rho\partial_-)$.

In writing (56) and (57) we have extracted the constant factor $\ln e^{-\rho}$.

The computation gives

$$\frac{\delta F_1}{\delta\rho(\xi)} = -\frac{26}{24\pi}\partial_+\partial_-\rho \tag{58}$$

and

$$\frac{\delta S_1}{\delta\rho(\xi)} = -\frac{2}{24\pi}\partial_+\partial_-\rho \ . \tag{59}$$

Putting now together the formulas (58) and (59) we obtain

$$\frac{\Delta_{FP}}{\det(x_-\partial_- - \frac{Q}{2\sqrt{2}}\partial_-^2)} = exp\left[-\frac{24}{24\pi}\int d^2\xi[\frac{1}{2}(\partial_+\rho\partial_-\rho) + \mu^2 e^{\rho}]\right] \tag{60}$$

where in the last expression we explicitly include the cosmological constant term.

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